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Rational solutions of the discrete time Toda lattice and the alternate discrete Painlevé II equation

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Abstract

The Yablonskii–Vorob'ev polynomials $y_n(t)$, which are defined by a second-order bilinear differential–difference equation, provide rational solutions of the Toda lattice. They are also polynomial tau-functions for the rational solutions of the second Painlevé equation (P_{II}). Here we define two-variable polynomials $Y_n(t, h)$ on a lattice with spacing h, by considering rational solutions of the discrete time Toda lattice as introduced by Suris. These polynomials are shown to have many properties that are analogous to those of the Yablonskii–Vorob'ev polynomials, to which they reduce when $h = 0$. They also provide rational solutions for a particular discretization of P_{II}, namely the so-called *alternate discrete* P_{II}, and this connection leads to an expression in terms of the Umemura polynomials for the third Painlevé equation (P_{III}). It is shown that the Bäcklund transformation for the alternate discrete Painlevé equation is a symplectic map, and the shift in time is also symplectic. Finally we present a Lax pair for the alternate discrete P_{II}, which recovers Jimbo and Miwa's Lax pair for P_{II} in the continuum limit $h \to 0$.

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1. Introduction

The Toda lattice

$$\frac{d^2 x_n}{dt^2} = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \qquad n \in \mathbb{Z} \tag{1}$$

was the first integrable differential–difference equation to be discovered [1]. The Yablonskii–Vorob'ev polynomials [2, 3] yield rational solutions of both the Toda lattice and the second Painlevé transcendent (P_{II}), since the tau-functions of P_{II} satisfy the bilinear form of the Toda lattice. In a previous work [4] one of the authors obtained an expression for solutions of the Toda lattice as ratios of Hankel determinants, by using the associated Lax pair to construct continued fraction solutions to a sequence of Riccati equations. This in turn led to an expression for the Yablonskii–Vorob'ev polynomials as Hankel determinants [5], equivalent to that discovered more recently [6] (see also [7]).

Here we will consider the case when the time evolution becomes discrete. Our approach is to start from the Lax pair for the discrete Toda lattice (dTL) given by Suris [8]. In section 2 we present this construction and derive polynomials $Y_n(t,h)$ in two variables t, h, which satisfy relations at a discrete set of points $t = t_0 + mh, m \in \mathbb{Z}$ (where t_0 is arbitrary). These polynomials tend to the Yablonskii–Vorob'ev polynomials $y_n(t)$ as the spacing $h \to 0$, so that $Y_n(t,0) = y_n(t)$. A discrete analogue of the bilinear defining equation for the y_n is derived and a corresponding representation of these Y_n as Hankel determinants is given.

The first few Yablonskii–Vorob'ev polynomials are

$$\begin{aligned} &y_0 = 1, \qquad y_1 = t, \qquad y_2 = t^3 + 4, \\ &y_3 = t^6 + 20t^3 - 80, \qquad y_4 = t^{10} + 60t^7 + 11200t. \end{aligned} \tag{2}$$

A further property of these polynomials, investigated recently by Clarkson and Mansfield [9], is the distribution of their zeroes. It is known that each $y_n(t)$ has no zeroes in common with $y_{n\pm1}(t)$, and that these zeroes are simple [10, 11]. Numerical studies indicate that these zeroes lie in approximately triangular arrays and that the zeroes of $y_n(t)$ interlace, in a certain sense, with those of $y_{n+1}(t)$, in a similar way to the zeroes of classical orthogonal polynomials. In section 3 we show that the polynomials $Y_n(t,h)$ have analogous properties.

In the continuum case the bilinear differential–difference equation that defines the Yablonskii–Vorob'ev polynomials is given in terms of the Hirota derivative D_t by

$$y_{n+1}y_{n-1} = ty_n^2 - 2D_t^2 y_n \cdot y_n, \tag{3}$$

with the initial polynomials $y_0 = 1$, $y_1 = t$, and this equation follows from the representation of P_{II} as a pair of bilinear equations for the associated tau-functions. In section 4 we consider a discretization of P_{II}, namely the alternate discrete P_{II} equation (alt-dP_{II}) studied in [12], and explain how its solutions are specified by tau-functions which satisfy a discrete bilinear equation together with a quadrilinear (degree four) relation. The bilinear relation defining the polynomials $Y_n(t,h)$ is a consequence of these tau-function equations. The alt-dP_{II} equation has Bäcklund transformations corresponding to the shifts $n \to n \pm 1$, just as in the continuum case, which imply that the discrete Yablonskii–Vorob'ev polynomials satisfy recurrence relations which are analogous to those in the continuum setting. It is also known that the alt-dP_{II} equation arises as the contiguity relations for a Bäcklund transformation for P_{III}. The latter connection leads to an alternative formula for the discrete Yablonskii–Vorob'ev polynomials in terms of determinants of Jacobi–Trudi type, corresponding to Umemura polynomials [13].

It is known from the work of Okamoto that the continuum P_{II} can be written in Hamiltonian form, and the Bäcklund transformation is a canonical transformation [14]. In section 5 we present a Poisson structure for the discrete case such that both development in time and the Bäcklund transformation for alt-dP_{II} are symplectic maps. Finally in section 6 we present a Lax pair for alt-dP_{II}, which relates it to the isomonodromic deformation of an associated linear system, and show that this tends to Jimbo and Miwa's Lax pair for the continuum case as $h \to 0$.

2. Discrete time Toda lattice and discrete Yablonskii–Vorob'ev polynomials

The equations for the discrete Toda lattice obtained by Suris [8] are

$$\begin{aligned} x_n(t+h) - x_n(t) &= h\pi_n(t+h), \\ \pi_n(t+h) - \pi_n(t) &= \frac{1}{h}\log\left[\frac{1+h^2\exp(x_{n-1}(t)-x_n(t))}{1+h^2\exp(x_n(t)-x_{n+1}(t))}\right], \end{aligned} \tag{4}$$

where π_n denotes the canonically conjugate momentum to x_n, and these discrete equations clearly yield Hamilton's equations for the Toda lattice (1) in the continuum limit, as $h \to 0$. They arise from the consistency condition for the Lax pair

$$\Psi_n(t+h) = \mathbf{V}_n(t)\Psi_n(t), \qquad \Psi_{n+1}(t) = \mathbf{L}_n(t)\Psi_n(t), \tag{5}$$

that is $\mathbf{L}_n(t+h)\mathbf{V}_n(t) = \mathbf{V}_{n+1}(t)\mathbf{L}_n(t)$, where

$$\mathbf{L}_n(t) = \begin{pmatrix} \lambda\exp(-h\pi_n(t)) - \frac{1}{\lambda} & h\exp(-x_n(t)) \\ -h\exp(x_n(t)-h\pi_n(t)) & 0 \end{pmatrix} \tag{6}$$

and

$$\mathbf{V}_n(t) = \begin{pmatrix} \frac{1}{\lambda} & -h\exp(-x_n(t)) \\ h\exp(x_{n-1}(t)) & \lambda \end{pmatrix}. \tag{7}$$

Upon setting $\Psi_n(t) = [X_n(t), Y_n(t)]^T$ and $Z_n(t) = X_n(t)Y_n(t)^{-1}$, we find from the second of (5) that

$$Z_n(t) = -\frac{h\exp(-x_n(t))}{\lambda\exp(-h\pi_n(t)) - \frac{1}{\lambda} + h\exp(-h\pi_n(t)+x_n(t))Z_{n+1}(t)}. \tag{8}$$

This recurrence relation can be used to generate the continued fraction expansion

$$\lambda h Z_0(t) = \frac{\gamma f_1}{1+\gamma g_1} + \frac{\gamma f_2}{1+\gamma g_2} + \frac{\gamma f_3}{1+\gamma g_3} + \cdots, \tag{9}$$

where $\gamma = \lambda^2$, $f_1 = h^2\exp(-x_0(t))$, $g_1 = -\exp(-h\pi_0(t))$ and for $n = 2, 3, \ldots$ we have

$$\begin{aligned} f_n &= -h^2\exp(-x_{n-1}(t) + x_{n-2}(t) - h\pi_{n-2}(t)), \\ g_n &= -\exp(-h\pi_{n-1}(t)). \end{aligned} \tag{10}$$

Similarly from the first of (5) we have the discrete time Riccati equation

$$Z_0(t+h) - Z_0(t) = -\frac{h}{\lambda}\,\mathrm{e}^{-x_0(t)} + \left(\frac{1}{\lambda^2} - 1\right)Z_0(t) - \frac{h}{\lambda}\,\mathrm{e}^{x_{-1}(t)}Z_0(t+h)Z_0(t). \tag{11}$$

In our previous work [15] we took $x_{-1}(t) \to -\infty$ so that this became a linear equation. Here we take $x_{-1}(t) = 0$ and set $Z_0(t) = \frac{1}{h\lambda}W(t)$ so that with $\gamma = \lambda^2$ we find

$$W(t+h) = -h^2\,\mathrm{e}^{-x_0(t)} + \frac{1}{\gamma}W(t) - \frac{1}{\gamma}W(t+h)W(t). \tag{12}$$

Now the continued fraction (9) has expansions both in positive and negative powers of γ, namely $W(t) = \sum_{n=1}^{\infty}\alpha_n(t)\gamma^n$ and $W(t) = \sum_{-\infty}^{n=0}\alpha_n(t)\gamma^n$ respectively. Substituting these expansions in turn into (12) and equating coefficients of corresponding powers of γ, we obtain the recurrence relations for their coefficients given respectively by

$$\alpha_{j+1}(t) = \alpha_j(t+h) + \sum_{k=1}^{j}\alpha_k(t+h)\alpha_{j-k+1}(t), \qquad j = 1, 2, 3, \ldots, \tag{13}$$

$$\alpha_j(t+h) = \alpha_{j+1}(t) - \sum_{k=j+1}^{0} \alpha_k(t+h)\alpha_{j+1-k}(t), \qquad j = -1, -2, \ldots \tag{14}$$

where $\alpha_1(t) = h^2 \exp(-x_0(t))$ and $\alpha_0(t+h) = -h^2 \exp(-x_0(t))$.

The continued fraction in (9) is known as a *T-fraction* [16] and its elements are given in terms of Hankel determinants. If we take the definitions

$$u_n = H_n^{(-n+2)}, \qquad v_n = H_n^{(-n+1)}, \tag{15}$$

with Hankel determinants

$$H_k^{(m)} = \begin{vmatrix} \beta_m & \beta_{m+1} & \cdots & \beta_{m+k-1} \\ \beta_{m+1} & \beta_{m+2} & \cdots & \beta_{m+k} \\ \vdots & \vdots & & \vdots \\ \beta_{m+k-1} & \beta_{m+k} & \cdots & \beta_{m+2k-2} \end{vmatrix} \tag{16}$$

whose elements are given by

$$\begin{aligned} \beta_k &= -\alpha_k(t), \qquad k = 1, 2, \ldots, \\ &= \alpha_k(t), \qquad k = 0, -1, -2, \ldots, \end{aligned} \tag{17}$$

then the elements of the T-fraction are given by

$$f_n = -\frac{v_{n-2}u_n}{v_{n-1}u_{n-1}}, \qquad g_n = -\frac{v_{n-1}u_n}{v_n u_{n-1}}, \qquad n = 2, 3, 4, \ldots. \tag{18}$$

From the expression for f_n, g_n in terms of the coordinates and momenta given by (10) and the equation of motion (4), it follows that

$$\frac{f_1(t+h)}{f_1(t)} = -g_1(t+h), \qquad \frac{g_1(t+h)}{g_1(t)} = \frac{1 + \frac{f_2(t)}{g_1(t)}}{1 + f_1(t)} \tag{19}$$

and more generally

$$\frac{f_{n+1}(t+h)}{f_{n+1}(t)} = \frac{g_{n+1}(t+h)}{g_n(t)}, \qquad \frac{g_{n+1}(t+h)}{g_{n+1}(t)} = \frac{1 + \frac{f_{n+2}(t)}{g_{n+1}(t)}}{1 + \frac{f_{n+1}(t)}{g_n(t)}}, \qquad n \in \mathbb{N}. \tag{20}$$

Proposition 2.1. *The two types of Hankel determinant defined by (15) are related by*

$$v_n(t+h) = u_n(t), \qquad n = 0, 1, 2, \ldots. \tag{21}$$

Proof. Substituting the expressions for f_n, g_n given by (18) into the first of (20) one finds that

$$\frac{v_{n+1}(t+h)u_n(t)}{v_n(t+h)u_{n+1}(t)} = \frac{v_1(t+h)u_0(t)}{v_0(t+h)u_1(t)}. \tag{22}$$

But $u_0 = H_0^{(2)} = 1 = H_0^{(1)} = v_0$ and $u_1(t) = \beta_1 = -h^2 \exp(-x_0(t)), v_1(t) = \beta_0 = -h^2 \exp(-x_0(t-h))$. Therefore the right-hand side of (22) equals unity and the result follows. □

Proposition 2.2. *When* $\exp(-x_0(t)) = -\frac{t}{4}$, *the Hankel determinants* u_n *satisfy the bilinear relation*

$$\left(1 - \frac{h^2 t}{4}\right) u_n(t+h)u_n(t-h) = u_n(t)^2 + u_{n+1}(t)u_{n-1}(t) \tag{23}$$

for $n = 1, 2, \ldots$.

Proof. Substituting for g_n and f_n from (18) into the second of (20) and then eliminating the dependence on the v_n using (21), the result follows from the fact that $u_1(t) = v_1(t+h) = \frac{h^2 t}{4}$ and $u_2(t) = -\frac{h^6}{64}(t^3 - h^2 t + 4)$. □

Remark. In [17], Hankel determinant solutions are given for a different (but gauge equivalent) bilinear form of the discrete Toda lattice equation, namely the equation $\rho_n^{l+1}\rho_n^{l-1} - (\rho_n^l)^2 = \varepsilon^2 \rho_{n+1}^{l+1}\rho_{n-1}^{l-1}$. Upon setting $\varepsilon^2 = 1$ and $t = (l-n)h$, $u_n(t) = \rho_n^{n+ht}/\rho_0^{ht}$ satisfies (23) for a suitable choice of the initial condition ρ_0^{ht} for $n = 0$.

It follows from the recurrence relations (14) that when $\exp(-x_0(t)) = -\frac{t}{4}$ the $\alpha_j(t)$, and hence also $u_n(t)$ and $v_n(t)$, are polynomials in h, t. We now renormalize u_n so that they are $\mathcal{O}(h^0)$ as $h \to 0$.

Definition 2.1. *The discrete Yablonskii–Vorob'ev polynomials $Y_n(t,h)$ are defined by the bilinear recurrence*

$$h^2 Y_{n+1}(t,h)Y_{n-1}(t,h) = (h^2 t - 4)Y_n(t+h,h)Y_n(t-h,h) + 4Y_n(t,h)^2 \tag{24}$$

for $n \geqslant 1$, with $Y_0(t,h) = 1$, $Y_1(t,h) = t$ as initial data.

Theorem 2.1. *The discrete Yablonskii–Vorob'ev polynomials $Y_n(t,h) \in \mathbb{Z}[t,h^2]$ are given in terms of the Hankel determinants u_n in (15) by $Y_n(t,h) = (-1)^n\left(-\frac{h^2}{4}\right)^{-n(n+1)/2}u_n(t)$ for $n = 0, 1, 2, \ldots$. Each Y_n is a monic polynomial of degree $n(n+1)/2$ in t, satisfying $Y_n(t,h) = y_n(t) + \mathcal{O}(h^2)$ as $h \to 0$, where $y_n(t)$ are the usual Yablonskii–Vorob'ev polynomials defined by (3) with $y_0 = 1$, $y_1 = t$.*

Proof. With $\alpha_1(t) = -h^2 t/4$ it is easy to prove by induction from the first recursion in (14) that $\alpha_j(t) = h^2 P_j/4^j$ for $j \geqslant 1$, where $P_j \in \mathbb{Z}[t,h]$. Similarly, with $\alpha_0(t) = h^2(t-h)/4$ the second recursion in (14) implies that $\alpha_j(t) = h^2 \hat{P}_j/4^{-j+1}$ for $j \leqslant 0$, where $\hat{P}_j \in \mathbb{Z}[t,h]$. It follows from their definition in terms of the matrix elements (17) that the Hankel determinants u_n are polynomials in $\mathbb{Q}[t,h]$, with powers of 4 as the only possible denominators of the coefficients. If we let $Y_n(t,h) = (-1)^n\left(-\frac{h^2}{4}\right)^{-n(n+1)/2}u_n(t)$ for $n \in \mathbb{N}$ then we find $Y_0(t,h) = 1$, $Y_1(t,h) = t$, and then the relation (24) follows from proposition 2.2, by substituting for $u_n(t)$ in (23) in terms of $Y_n(t,h)$. Since the Y_n are uniquely defined by the recurrence (24) together with the given initial data, the formula in terms of renormalized Hankel determinants guarantees that they are polynomials in t, and the recurrence also implies that they are monic and of degree $d_n := n(n+1)/2)$ in t. However, further analysis is required to verify that there are no powers of h or powers of 4 in the denominator for this choice of normalization.

By induction, suppose that $Y_n(t,h) \in \mathbb{Z}[t,h]$, and hence is regular as $h \to 0$, for $n = 0, 1, \ldots, N$ (which clearly holds for $N = 1$). Taking a Taylor expansion in t, with derivatives denoted by $Y_{N,(j)t} = \frac{\partial^j Y_N}{\partial t^j}$, we see that

$$\begin{aligned} Y_N(t+h,h)Y_N(t-h,h) &= \left(\sum_{k=0}^{[d_N/2]} \frac{h^{2k}Y_{N,(2k)t}(t,h)}{(2k)!}\right)^2 - h^2\left(\sum_{k=0}^{[(d_N-1)/2]} \frac{h^{2k}Y_{N,(2k+1)t}(t,h)}{(2k+1)!}\right)^2 \\ &= Y_N(t,h)^2 + h^2\tilde{P}(t,h), \end{aligned}$$

where $\tilde{P}(t,h)$ is a polynomial in t and h by the inductive hypothesis. Moreover, at leading order we have

$$\tilde{P}(t,h) = Y_N(t,0)Y_{N,tt}(t,0) - Y_{N,t}(t,0)^2 + \mathcal{O}(h).$$

Substituting this into the right-hand side of (24) and dividing by h^2 we have

$$Y_{N+1}(t,h)Y_{N-1}(t,h) = tY_N(t+h,h)Y_N(t-h,h) - 4\tilde{P}(t,h), \tag{25}$$

from which it follows that $Y_{N+1}(t,h)$ is regular as $h \to 0$, and hence lies in $\mathbb{Q}[t,h]$ with at worst powers of 4 as denominators of its coefficients. Now for some $K \geqslant 0$ we can write $Y_{N+1}(t,h) = \hat{Y}_{N+1}(t,h)/4^K$ where $\hat{Y}_{N+1} \in \mathbb{Z}[t,h]$ with $4 \nmid \hat{Y}_{N+1}$. If we multiply both sides of (24) by 4^K we see that if $K > 0$ then we have $4|h^2\hat{Y}_{N+1}Y_{N-1}$, but then since Y_{N-1} is monic in t it is clear that for $2 \nmid Y_{N-1}(t,h)$ we must have $4|\hat{Y}_{N+1}$, which is a contradiction. Hence $K = 0$ and $Y_{N+1} \in \mathbb{Z}[t,h]$ as required.

Setting $h \to -h$ in the recurrence (24) it also follows immediately by induction that $Y_n(t,h) = Y_n(t,-h)$ for all n, so in fact we have polynomials in $\mathbb{Z}[t,h^2]$. Thus we can write $Y_n(t,h) = y_n(t) + \mathcal{O}(h^2)$ where $y_0(t) = Y_0(t,h) = 1$ and $y_1(t) = Y_1(t,h) = t$, and using the leading order part of $\tilde{P}(t,h)$ in (25) we find that for $n \geqslant 1$ y_n satisfy

$$y_{n+1}y_{n-1} = ty_n^2 - 4y_n\ddot{y}_n + 4\dot{y}_n^2,$$

which is precisely the defining relation (3) for the usual Yablonskii–Vorob'ev polynomials. □

The first few discrete Yablonskii–Vorob'ev polynomials are

$$\begin{aligned}
Y_0(t,h) &= 1,\\
Y_1(t,h) &= t,\\
Y_2(t,h) &= t^3 + 4 - h^2 t,\\
Y_3(t,h) &= t^6 + 20t^3 - 80 + h^2(4t - 5t^4) + 4t^2h^4,\\
Y_4(t,h) &= t^{10} + 60t^7 + 11200t - h^2(15t^8 + 252t^5 + 3360t^2)\\
&\quad + h^4(63t^6 + 480t^3 + 576) - h^6(85t^4 + 288t) + 36h^8t^2.
\end{aligned} \tag{26}$$

These examples clearly reduce to the usual Yablonskii–Vorob'ev polynomials (2) when $h = 0$.

3. Zeroes of the discrete Yablonskii–Vorob'ev polynomials

In the continuum case it has been shown that the zeroes of $y_n(t)$ are simple and are not zeroes of $y_{n+1}(t)$ [10, 11]. Here we prove an analogous result in the discrete case, by considering the roots of $Y_n(t,h)$ as a polynomial in t, that is $t_0 = t_0(h)$ such that $Y_n(t_0(h),h) = 0$. To begin with we require a simple observation.

Lemma 3.1. *The discrete Yablonskii–Vorob'ev polynomials never vanish at $t = 4/h^2$. More precisely,*

$$Y_n\left(\frac{4}{h^2}, h\right) = \left(\frac{4}{h^2}\right)^{n(n+1)/2} \neq 0 \qquad \text{for all} \quad n \in \mathbb{N}.$$

Proof. The result is trivially true when $n = 0, 1$. Assume that it holds for $n = 1, 2, \ldots, N$. Then from (24) $h^2Y_{N+1}(4/h^2,h)Y_{N-1}(4/h^2,h) = 4Y_N(4/h^2,h)^2 \neq 0$, and the result follows by induction. □

From theorem 2.1 we know that $Y_n(t+h,h) - Y_n(t,h) = h\dot{y}_n(t) + \mathcal{O}(h^2)$, and the zeroes of $Y_n(t,h)$ in t differ from those of $y_n(t)$ by $\mathcal{O}(h^2)$, so (because the zeroes of y_n are simple) we expect that $Y_n(t+h,h)$ should not vanish where $Y_n(t,h)$ does. Indeed this turns out to be the case. In the following section we shall see that the polynomials Y_N are tau-functions for a discrete PII equation, and hence satisfy many other bilinear identities in addition to (24). To prove the following result we make use of one such identity, namely (50) below.

Theorem 3.1. *For any $n = 1, 2, \ldots$, if t_0 is a zero of Y_n, so that $Y_n(t_0, h) = 0$, then $Y_n(t_0 \pm h, h) \neq 0$ and $Y_{n+1}(t_0, h) \neq 0$.*

Proof. It is true for $n = 1$ by inspection. Suppose the result is true for $n = 1, 2, \ldots, N-1$. Now assume that both $Y_N(t_0, h) = 0$ and $Y_N(t_0 + h, h) = 0$. Using the identity (50) with $\tau_n = Y_n(t, h)$, and setting $n = N-1$, $t = t_0$, we have

$$Y_N(t_0+h, h)Y_{N-2}(t_0, h) - Y_N(t_0, h)Y_{N-2}(t_0+h, h) \\ = (2N-1)hY_{N-1}(t_0+h, h)Y_{N-1}(t_0, h).$$

By the assumption, the left-hand side vanishes, and $2N - 1 \neq 0$ for $N \in \mathbb{Z}$ which means that $Y_{N-1}(t_0, h)$ or $Y_{N-1}(t_0 + h, h) = 0$, but this contradicts the inductive hypothesis, so $Y_N(t_0 + h, h) \neq 0$ whenever $Y_N(t_0, h) = 0$ (and similarly for $Y_N(t_0 - h, h)$). Now if $Y_N(t_0, h) = 0$, then from (24) we see that

$$h^2 Y_{N-1}(t_0, h)Y_{N+1}(t_0, h) = (h^2 t_0 - 4)Y_N(t_0+h, h)Y_N(t_0-h, h)$$

If we now also assume that $Y_{N+1}(t_0, h) = 0$, then the left-hand side of the above vanishes, and since $h^2 t_0 - 4 \neq 0$ by lemma 3.1 then $Y_N(t_0 + h, h) = 0$ or $Y_N(t_0 - h, h) = 0$, which is a contradiction. Therefore the result holds for $n = N$, hence for all n by induction. □

Clarkson and Mansfield have numerically studied the location of zeroes of $y_n(t)$ for low values of n [9]. They found that for a given n they occupy approximate triangular arrays in the complex plane and that the zeroes of $y_n(t)$ *interlace* in a certain sense with those of $y_{n+1}(t)$. Since the zeroes of $Y_n(t, h)$ are the same as those of $y_n(t)$ up to $\mathcal{O}(h^2)$, the same picture holds for sufficiently small h, and our numerical observations show that the same qualitative behaviour persists for values of h up to at least order of unity.

4. Alternate discrete Painlevé II

The second Painlevé equation (P_{II}) is

$$\ddot{q} = 2q^3 + tq + \alpha, \tag{27}$$

where α is a constant. This second-order differential equation is equivalent to a first-order system introduced by Okamoto [14], namely

$$\begin{aligned} \dot{q} &= p - q^2 - \frac{t}{2} \\ \dot{p} &= 2qp + \ell, \qquad \ell = \alpha + \tfrac{1}{2}. \end{aligned} \tag{28}$$

Rational solutions of (27) arise when $\alpha = n \in \mathbb{Z}$. However, in this and subsequent sections n need not be an integer unless stated explicitly otherwise, and we use the parameters n and α interchangeably. We also find it convenient to use the shifted parameter $\ell = n + 1/2$ as above, which fixes a point in the A_1 root space. This is the parameter used by Okamoto to label solutions of P_{II} and the corresponding tau-functions, but we stick with the label n to maintain contact with the preceding results. P_{II} may be put into the bilinear form by setting

$$q_n = \frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{\tau_{n-1}(t)}{\tau_n(t)}\right). \tag{29}$$

Then (27) with $\alpha = n$ is equivalent to the bilinear equations

$$\begin{aligned} &D_t^2 \tau_n \cdot \tau_{n-1} = F(t)\tau_n\tau_{n-1}, \\ &\left(D_t^3 - tD_t + n\right)\tau_n \cdot \tau_{n-1} = 3F(t)D_t\tau_n \cdot \tau_{n-1}, \end{aligned} \tag{30}$$

where D_t is the usual Hirota operator, and the function $F(t)$ is arbitrary. By a gauge transformation on the tau-functions, that is $\tau_n \to G(t)\tau_n$ for all n, the function F can be set to zero in (30) without loss of generality.

The Bäcklund transformation linking a solution of (27) for $\alpha = n$ with that for $\alpha = n+1$ is

$$q_{n+1} = -q_n - \frac{\ell}{p_n}, \qquad p_n = \dot{q}_n + q_n^2 + t/2, \tag{31}$$

and the two solutions are related by

$$\dot{q}_n + q_n^2 = -\dot{q}_{n+1} + q_{n+1}^2 \tag{32}$$

(which can be seen as being inherited from the Miura transformation for KdV). From (30) and (31) it may be proved that, with the same choice of gauge that fixes $F \equiv 0$ in (30), any three adjacent tau-functions $\tau_n, \tau_{n\pm1}$ satisfy the bilinear relations

$$D_t \tau_{n+1} \cdot \tau_{n-1} = 2\ell\tau_n^2 \tag{33}$$

and

$$\tau_{n+1}\tau_{n-1} = t\tau_n^2 - 2D_t^2\tau_n \cdot \tau_n. \tag{34}$$

With this choice of gauge, equation (34) is the bilinear form of the Toda lattice. As we have already mentioned (cf equation (3) in the introduction), the latter relation is the defining recurrence for the Yablonksii–Vorob'ev polynomials; with $\tau_n = y_n$ they satisfy the other bilinear equations (33) and (30) for $F = 0$, and hence determine the rational solutions of P_{II}.

Several discretizations of P_{II} have been studied in recent years including a discrete bilinear form [18] which tends to (30) in the continuum limit. The discrete bilinear form allowed the construction of rational solutions to this discrete P_{II}, given in terms of ratios of determinants of Jacobi–Trudi type [19], while solutions in terms of discrete Airy functions were found in [20, 21]. A hierarchy of higher order analogues of this discrete P_{II} (dP_{II}) have been given by Cresswell and Joshi, based on a Lax pair. However, due to the non-uniqueness of the discretization process, there are many other $dPII$ equations with analogous properties. For example, four different dP_{II} equations are mentioned in [23], three of which are q-type (q-P_{II}). Discrete Airy function solutions of one such q-P_{II} equation are found in [24], while another one is considered with its ultra-discretization in [25], along with yet one more distinct q-P_{II}. In [26] a detailed study of a q-P_{II} in Sakai's classification [27] has been performed. Another alternative discretization, which is relevant here, is the so-called *alternate* dP_{II} (alt-dP_{II}) which was studied in [12], and appeared in connection with the original dP_{II} in [28].

Here we consider the alt-dP_{II} equation in the form of the second-order difference equation

$$\frac{\frac{h^3}{2}m - 2}{\overline{g}g + 1} + \frac{\frac{h^3}{2}(m-1) - 2}{g\underline{g} + 1} = -g + \frac{1}{g} + \frac{h^3}{2}(m-\ell) - 2 \tag{35}$$

with the notation $g = g_n(t)$, $\overline{g}_n = g_n(t+h, h)$, $\underline{g}_n = g_n(t-h, h)$.[1] This is referred to as a discrete P_{II} equation due to the fact that (27) arises from it by setting $g(t,h) = -1 + hq(t,h)$ and taking the continuum limit $h \to 0$ (with the parameter $\alpha = \ell - 1/2$ as usual). If we set $g_n(t,h) = x_m$ with $z_m = h^3m/2 - 2$ and $\mu = -h^3\ell/2$ then this is equation (1.3) in [12] (except that we have m in place of n).

[1] The introduction of m in place of t is not essential, but it suggests that m and ℓ might be considered on the same footing, so that $w_{\ell,m} \equiv g_n(t)$ should provide particular solutions of a suitable partial difference equation. See [12] for a connection with the lattice modified BSQ equation.

The $alt\text{-}dP_{II}$ equation (35) has rational solutions for integer values of the parameter $n = \ell - 1/2$, provided by suitable ratios of the discrete Yablonskii–Vorob'ev polynomials $Y_n(t, h)$. In order to prove this, we must derive appropriate discrete analogues of the bilinear identities (30), (33) and (34). We start by presenting (35) in the form of a system, which makes it more manageable.

Lemma 4.1. *The $alt\text{-}dP_{II}$ equation (35) for $\ell = n+\frac{1}{2}$, $m = t/h$ is equivalent to the first-order system*

$$\begin{aligned} \overline{g}_n &= (-2 + h^2 p_n)/(4 + 2g_n - h^2(g_n p_n + t)), \\ \overline{p}_n &= (p_n - \ell h \overline{g}_n)/\overline{g}_n^2. \end{aligned} \tag{36}$$

Proof. By rearranging the first equation in (36) we have

$$p_n = \frac{1}{h^2}\left(\frac{(4 - h^2 t)\overline{g}_n}{1 + g_n \overline{g}_n} + 2\right), \tag{37}$$

and by shifting $t \to t + h$ the latter provides a formula for $\overline{p}_n$ in terms of $\overline{g}_n$ and $\overline{\overline{g}}_n$, and then substituting this into the left-hand side of the second equation (36), with p_n given by (37) on the right-hand side, gives a relation between g_n, $\overline{g}_n$ and $\overline{\overline{g}}_n$. After rearranging, shifting $t \to t - h$ and setting $t = mh$, this relation is precisely (35). Conversely, given the $alt\text{-}dPII$ equation (35), we can define p_n by the formula (37), which is equivalent to the first equation in (36); the second equation for $\overline{p}_n$ then follows immediately from $alt\text{-}dPII$. □

The Bäcklund transformation for the $alt\text{-}dP_{II}$ equation (35) is most easily derived starting from the analogue of (32), and the following results are easily verified by direct calculation.

Lemma 4.2. *If g_{n+1} is given by*

$$g_{n+1} = \frac{p_n}{\ell h + g_n p_n}, \tag{38}$$

in terms of g_n and p_n satisfying (36), then the identity

$$\frac{1}{\underline{g}_n} + g_n = \frac{1}{g_{n+1}} + \underline{g}_{n+1} \tag{39}$$

holds.

Corollary 4.1. *The quantity g_{n+1}, defined by (38) with p_n given by (37), satisfies (35) with $\ell \to \ell + 1$. Equivalently, equation (38) and the relation*

$$p_{n+1} = \frac{1}{h^2}\left(\frac{(4 - h^2 t)\overline{g}_{n+1}}{1 + g_{n+1}\overline{g}_{n+1}} + 2\right) \tag{40}$$

together constitute a Bäcklund transformation for the $alt\text{-}dP_{II}$ system (36).

Corollary 4.2. *The Bäcklund transformation for the $alt\text{-}dP_{II}$ system, given by the formulae (38) and (40), has the following consequences:*

$$g_{n+1}\underline{p}_n = g_n p_n; \tag{41}$$

$$\frac{1}{\overline{g}_n} - \overline{g}_{n+1} = \frac{\ell h}{p_n}; \tag{42}$$

$$\overline{g}_{n+1} g_n - \frac{1}{g_{n+1}\overline{g}_n} = -\frac{\ell h}{p_n}\left(g_n + \frac{1}{\overline{g}_n}\right). \tag{43}$$

Remark. With $g_n = -1 + hq_n$, the identity (32) arises as the continuum limit of (39), and the formula (31) arises from (38), as $h \to 0$. Similarly, the system (28) is the continuum limit of (36). Nijhoff *et al* derived equivalent formulae of Miura/Schlesinger type for the Bäcklund transformation of $alt\text{-}dP_{II}$ by making use of a variable y_n (see equation (5.1) in [12]), which (modulo rescaling and replacing n by m) is analogous to p_n defined by (37).

We now describe the tau-functions for the $alt\text{-}dP_{II}$ equation, which satisfy analogues of (30).

Proposition 4.1. *Up to a choice of gauge, every solution of (35) is specified by a pair of tau-functions* $\tau_n(t,h)$, $\tau_{n-1}(t,h)$ *via the formula*

$$g_n(t,h) = -\frac{\tau_{n-1}(t-h,h)\tau_n(t,h)}{\tau_{n-1}(t,h)\tau_n(t-h,h)}, \tag{44}$$

where the tau-functions satisfy the bilinear equation

$$\overline{\tau}_n\underline{\tau}_{n-1} + \underline{\tau}_n\overline{\tau}_{n-1} = 2\tau_n\tau_{n-1} \tag{45}$$

and the quadrilinear (degree four) equation

$$\begin{aligned}(4 - mh^3)\underline{\tau}_{n-1}\underline{\tau}_n(\overline{\tau}_{n-1}\underline{\tau}_n - \underline{\tau}_{n-1}\overline{\tau}_n) + (4 - (m-1)h^3)\tau_{n-1}\tau_n(\tau_{n-1}\underline{\underline{\tau}}_n - \underline{\underline{\tau}}_{n-1}\tau_n)\\ + 8\big(\underline{\tau}_{n-1}^2\tau_n^2 - \tau_{n-1}^2\underline{\tau}_n^2\big) - 4nh^3\underline{\tau}_{n-1}\underline{\tau}_n\tau_{n-1}\tau_n = 0,\end{aligned} \tag{46}$$

with $m = t/h$, $n = \alpha = \ell - 1/2$ *and* $\tau_n = \tau_n(t,h)$, $\overline{\tau}_n = \tau_n(t+h,h)$, $\underline{\tau}_n = \tau_n(t-h,h)$, *etc.*

Proof. Upon substituting the tau-function expression (44) into (35) and clearing denominators, a relation of degree eight results, which can be simplified somewhat by rewriting it in terms of the symmetric/antisymmetric quadratic quantities $A_\pm = \overline{\tau}_{n-1}\underline{\tau}_n \pm \underline{\tau}_{n-1}\overline{\tau}_n$ and $\underline{A}_\pm = \tau_{n-1}\underline{\underline{\tau}}_n \pm \underline{\underline{\tau}}_{n-1}\tau_n$. In general, for any choice of tau-functions the bilinear equation

$$\overline{\tau}_n\underline{\tau}_{n-1} + \underline{\tau}_n\overline{\tau}_{n-1} = 2\hat{F}\tau_n\tau_{n-1}$$

holds, for some function $\hat{F} = \hat{F}(t,h)$, but by applying a gauge transformation $\tau_n \to \hat{G}\tau_n$, $\tau_{n-1} \to \hat{G}\tau_{n-1}$ with $\overline{\hat{G}}\underline{\hat{G}}/\hat{G}^2 = \hat{F}$ the function $\hat{F}$ can be removed to yield the bilinear equation (45). With this choice of gauge, the remaining terms in the degree eight relation factorize to yield the quadrilinear equation (46), and conversely if these two tau-function equations hold then g_n given by (44) is a solution of (35) for $\ell = n + 1/2 = \alpha + 1/2$. □

Remark. The existence of a quadrilinear relation between a pair of tau-functions is mentioned in section 4 of [12], where a third tau-function is introduced to obtain purely bilinear relations (cf theorem 4.1 below).

It is easy to see that (45) tends to the first of (30) in the continuum limit (with the gauge chosen so that $F = 0$). Although the second relation (46) between the two tau-functions is of overall degree four, it still produces the second bilinear differential equation (30) in the continuum limit (provided that the first one also holds). In order to work with purely bilinear equations in the discrete case, we must consider three adjacent tau-functions τ_n, $\tau_{n\pm1}$.

Theorem 4.1. *Up to a choice of gauge, every solution of the* $alt\text{-}dP_{II}$ *system (36) is specified by three tau-functions* $\tau_{n-1}(t)$, $\tau_n(t)$, $\tau_{n+1}(t)$, *with* g_n *given by (44) and*

$$p_n = \frac{\tau_{n-1}\tau_{n+1}}{2\tau_n^2}, \tag{47}$$

where the tau-functions satisfy (45) as well as

$$\overline{\tau}_{n+1}\underline{\tau}_n + \underline{\tau}_{n+1}\overline{\tau}_n = 2\tau_{n+1}\tau_n \tag{48}$$

and

$$h^2\tau_{n+1}\tau_{n-1} = (h^2 t - 4)\overline{\tau}_n\underline{\tau}_n + 4\tau_n^2. \tag{49}$$

With this choice of normalization the identities

$$\overline{\tau}_{n+1}\tau_{n-1} - \tau_{n+1}\overline{\tau}_{n-1} = 2\ell h\overline{\tau}_n\tau_n \tag{50}$$

and

$$\overline{\tau}_{n+1}\underline{\tau}_{n-1} - \underline{\tau}_{n+1}\overline{\tau}_{n-1} = 4\ell h\tau_n^2 \tag{51}$$

also hold. For this choice of gauge, these purely bilinear relations are compatible with the Bäcklund transformation (38) for alt-dP_{II}, in the sense that τ_n and τ_{n+1} satisfy (46) for $n \to n+1$, and

$$g_{n+1}(t,h) = -\frac{\tau_n(t-h,h)\tau_{n+1}(t,h)}{\tau_n(t,h)\tau_{n+1}(t-h,h)}$$

satisfies (35) with $\ell \to \ell + 1$.

Proof. If a solution g_n, p_n of (36) is given by the expressions (44) and (47) respectively, and the gauge is fixed by (45), then the latter implies that

$$g_n + \frac{1}{\underline{g}_n} = -2\frac{\tau_n^2}{\overline{\tau}_n\underline{\tau}_n} = \frac{1}{g_{n+1}} + \overline{g}_{n+1}$$

by lemma 4.2, where g_{n+1} given by (38) is a solution of (35) with $\ell \to \ell+1$. The first equality above implies (49), while the relation (41) implies that g_{n+1} is given in terms of tau-functions by the formula (44) with $n \to n+1$, and hence (48) follows from the second equality above. The bilinear identities (50) and (51) then hold as a consequence of the relations (42) and (43) respectively. By proposition 4.1, the given choice of gauge implies that the pair τ_{n-1}, τ_n satisfy (46), and the fact that g_{n+1} is a solution of *alt-dPII* with the parameter shifted implies that the pair τ_n, τ_{n+1} also satisfy this quadrilinear equation with $n \to n+1$.

Conversely, suppose that g_n is defined in terms of tau-functions by (44), g_{n+1} is defined by the same relation for $n \to n+1$, and p_n is defined by (47), where the tau-functions satisfy the three relations (45), (48) and (49). The identities (37) and (39) follow immediately. Furthermore, from (45) and (48) it is clear that

$$\frac{\tau_{n-1}}{\tau_n}\left(\frac{\overline{\tau}_{n+1}}{\overline{\tau}_n} + \frac{\underline{\tau}_{n+1}}{\underline{\tau}_n}\right) = \frac{\tau_{n+1}}{\tau_n}\left(\frac{\overline{\tau}_{n-1}}{\overline{\tau}_n} + \frac{\underline{\tau}_{n-1}}{\underline{\tau}_n}\right),$$

which implies that

$$\frac{\overline{\tau}_{n+1}\tau_{n-1} - \tau_{n+1}\overline{\tau}_{n-1}}{\overline{\tau}_n\tau_n} = \frac{\tau_{n+1}\underline{\tau}_{n-1} - \underline{\tau}_{n+1}\tau_{n-1}}{\tau_n\underline{\tau}_n}.$$

Therefore $(\overline{\tau}_{n+1}\tau_{n-1} - \tau_{n+1}\overline{\tau}_{n-1})/(2h\overline{\tau}_n\tau_n)$ is independent of t, and if we denote this by ℓ, then we have the bilinear equation (50), which implies that (42) also holds. Solving (42) for $\overline{g}_{n+1}$ gives an expression in terms of p_n and $\overline{g}_n$, which in turn means that $\overline{g}_{n+1}$ can be written in terms of g_n and $\overline{g}_n$ using (37). By shifting $t \to t-h$, this gives a formula for g_{n+1} in terms of $\underline{g}_n$ and g_n, and then substituting for $\overline{g}_{n+1}$ and g_{n+1} in (39) yields the alt-dP_{II} equation (35). It then follows that g_n, p_n satisfy the system (36). □

Theorem 4.2. *For the parameter $\ell = n + \frac{1}{2}$ with $n \in \mathbb{Z}$ the alt-dP_{II} equation (35) has rational solutions given in terms of the discrete Yablonskii–Vorob'ev polynomials by*

$$g_n = -\frac{Y_{n-1}(t-h,h)Y_n(t,h)}{Y_{n-1}(t,h)Y_n(t-h,h)},$$

where the polynomials are extended to negative indices n by setting $Y_{-n} = Y_{n-1}$ *for* $n \in \mathbb{N}$. *As well as the defining recurrence (24), the relations (45), (46), (50) and (51) are satisfied by* $\tau_n(t, h) = Y_n(t, h)$ *for all* $n \in \mathbb{Z}$.

Proof. When $\ell = 1/2$, equation (35) has the trivial constant solution $g_0(t, h) = -1$, which can be obtained by setting $\tau_0 = Y_0 = 1 = Y_{-1} = \tau_{-1}$ in (44), and from (37) we have $p_0 = t/2$ which gives $\tau_1 = Y_1 = t$ by (47). It is easy to verify that each of the bilinear equations (45), (48) and (49) is satisfied by these tau-functions. By applying the Bäcklund transformation (38) repeatedly (both forwards and backwards) a doubly infinite sequence of rational solutions $\{g_n\}_{n\in\mathbb{Z}}$ is obtained. Then by theorem 4.1, since the Bäcklund transformation is compatible with the choice of gauge, it follows by induction that the corresponding tau-functions satisfy the identities (24), (45), (46), (50) and (51) for all $n \in \mathbb{Z}$. Since the Yablonskii–Vorob'ev polynomials are defined by (49) with $\tau_0 = 1$, $\tau_1 = t$, it follows that this particular sequence of tau-functions is given by $\tau_n(t, h) = Y_n(t, h)$ for all $n \in \mathbb{N}$. The fact that this relation can be consistently extended to negative n follows from the observation that all of the tau-function identities in proposition 4.1 and theorem 4.1 are invariant under $n \to -n - 1$, $\ell \to -\ell$. □

Remark. The simplest rational solutions of $alt\text{-}dP_{II}$ (corresponding to $n = 0, \pm 1$) are described in section 6 of the paper [12] by Nijhoff *et al* where it is indicated how the above sequence of rational solutions can be generated recursively via the Bäcklund transformation, but no closed form for these rational solutions is given in that work.

The fact that the $alt\text{-}dP_{II}$ equation can be derived from a sequence of Bäcklund transformations applied to solutions of P_{III} provides a relation between the discrete Yablonskii–Vorob'ev polynomials and the Umemura polynomials for P_{III}. The third Painlevé equation, P_{III}, is given by

$$w'' = \frac{(w')^2}{w} - \frac{w'}{x} + \frac{1}{x}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}, \tag{52}$$

with the prime ' denoting d/dx, and without loss of generality (by rescaling the independent variable x) if $\gamma\delta \neq 0$ the latter two parameters can be fixed as $\gamma = -\delta = 1$. (Note that the parameter α in (52) should not be confused with the parameter α in P_{II}.) Bäcklund transformations for P_{III} (cf section 2 in [12]) can be used to relate three adjacent solutions $w_n = w(x; \alpha, \beta)$ and $w_{n\pm1} = w(x; \alpha \pm 2, \beta \pm 2)$ and if we set $g_n = -1/w_n$ then this contiguity relation can be written in the form of the $alt\text{-}dP_{II}$ equation

$$\frac{z_n}{g_{n+1}g_n - 1} + \frac{z_{n-1}}{g_n g_{n-1} - 1} + \frac{x}{2}\left(g_n + \frac{1}{g_n}\right) + z_n + \mu = 0, \tag{53}$$

where

$$z_n = (\alpha + \beta + 2)/4, \qquad \Delta z_n := z_{n+1} - z_n = 1, \qquad \mu = (\beta - \alpha - 2)/4. \tag{54}$$

(To compare with equation (2.5) in [12], set $g_n = \mathrm{i}x_n$, $x = t$ in the above.) For a certain set of parameter values, P_{III} has rational solutions which are described by the following result.

Theorem 4.3. *(Kajiwara & Masuda [13]) For parameters*

$$\alpha = 2n + 2\nu - 1, \qquad \beta = 2n - 2\nu + 1, \qquad n \in \mathbb{Z} \tag{55}$$

and $\gamma = -\delta = 1$, *the third Painlevé equation (52) has rational solutions* $w(x; \alpha, \beta) = w_n$ *given by*

$$w_n = \frac{\mathcal{D}_n(x, \nu - 1)\mathcal{D}_{n-1}(x, \nu)}{\mathcal{D}_n(x, \nu)\mathcal{D}_{n-1}(x, \nu - 1)} \tag{56}$$

where the polynomial $\mathcal{D}_n$ is given by a determinant of Jacobi–Trudi type,

$$\mathcal{D}_n(x,\nu) = \begin{vmatrix} p_n & p_{n+1} & \cdots & p_{2n-1} \\ p_{n-2} & p_{n-1} & \cdots & p_{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ p_{-n+2} & p_{-n+3} & \cdots & p_1 \end{vmatrix}$$

with $p_k = p_k(x,\nu)$ defined by the generating function

$$\sum_{k=0}^{\infty} p_k(x,\nu)\lambda^k = (1+\lambda)^\nu \exp(x\lambda)$$

and $p_k = 0$ for $k < 0$.

Remarks. The polynomials $p_k(x,\nu)$ are essentially just associated Laguerre polynomials, and, for each n, $\mathcal{D}_n(x,\nu)$ is a Schur polynomial with restricted arguments, corresponding to the partition $(n, n-1, \ldots, 2, 1)$. The result stated above is an adapted form of theorem 1 in [13], and describes one family of rational solutions of P_{III}; for a complete description of all rational solutions of P_{III} for $\gamma\delta \neq 0$, see [29]. The polynomials $\mathcal{D}_n(x,\nu)$ (after some scaling) are known as the Umemura polynomials for P_{III}. Further properties of scaled Umemura polynomials for P_{III} are detailed in [30], including the remarkable patterns formed by the roots, and differential/difference equations; analogous polynomials corresponding to the special cases when $\gamma\delta = 0$ are also treated there.

Theorem 4.4. *The discrete Yablonskii–Vorob'ev polynomials are given in terms of determinants of Jacobi–Trudi type by the formula*

$$Y_n(t,h) = c_n h^{n(n+1)/2}\mathcal{D}_n\left(\frac{4}{h^3}, \frac{t}{h} - \frac{4}{h^3}\right), \tag{57}$$

where

$$c_n = (2n-1)!!(2n-3)!!\ldots 3!!1!! \tag{58}$$

for $n \in \mathbb{N}$.

The proof of the preceding theorem makes use of some results in the following section, and is relegated to the appendix. However, it is clear that if we rearrange the formula (57) then we can rewrite $\mathcal{D}_n(x,\nu)$ in the form of a Hankel determinant.

Corollary 4.3. *The Umemura polynomials for P_{III}, given in scaled form by $\mathcal{D}_n(x,\nu)$, are proportional to the Hankel determinants u_n as in (15) with $h = (x/4)^{-1/3}$ and $t = (x/4)^{-1/3}(x+\nu)$.*

Remark. It is known that the Painlevé differential equations form a coalescence cascade from P_{VI} down to P_I (see [31]). In [13] it is shown that the coalescence limit from P_{III} to P_{II} produces the Yablonskii–Vorob'ev polynomials $y_n(t)$ as a limit of the Umemura polynomials, but this arises in a different way compared with the limit $h \to 0$ considered above. More precisely, with the scaling used here, the coalescence from (52) to (27) arises when the independent variable x and the parameter ν scale as $x = \frac{t}{\epsilon} + \frac{4}{\epsilon^3}$, $\nu = \frac{1}{2} - \frac{4}{\epsilon^3}$, with $\epsilon \to 0$. In this limit, up to scaling the polynomials $\mathcal{D}_n(t/\epsilon + 4/\epsilon^3, 1/2 - 4/\epsilon^3)$ produce $y_n(t)$ at leading order in ϵ.

5. Symplectic properties and discrete P_{XXXIV}

Okamoto [14] showed that P_{II} can be written as the system (28) which is in Hamiltonian form, i.e.

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}, \tag{59}$$

with

$$H = \frac{p^2}{2} - \left(q^2 + \frac{t}{2}\right) p - \ell q. \tag{60}$$

Eliminating p gives (27), which is P_{II}, whilst eliminating q gives

$$\ddot{p} = \frac{\dot{p}^2}{2p} + 2p^2 - tp - \frac{\ell^2}{2p} \tag{61}$$

which is known as P_{XXXIV} (see [31], chapter XIV).

This representation has been used in [32] to study further properties of the Yablonskii–Vorob'ev polynomials. Although the alt-dP_{II} equation (35), being a non-autonomous difference equation, does not have a Hamiltonian form, many of the results proved there have their counterparts in the discrete case. For ease of comparison, we briefly recall some known results on P_{II}. In terms of the canonical coordinates (q_n, p_n), the Bäcklund transformation for P_{II} can be written (together with its inverse) as

$$\begin{aligned} q_{n+1} &= -q_n - \frac{\ell}{p_n}, \qquad p_{n+1} = -p_n + t + 2\left(q_n + \frac{\ell}{p_n}\right)^2; \\ q_{n-1} &= -q_n - \frac{(\ell - 1)}{2q_n^2 - p_n + t}, \qquad p_{n-1} = -p_n + t + 2q_n^2. \end{aligned} \tag{62}$$

It is straightforward to check that $dq_{n+1} \wedge dp_{n+1} + dH_{n+1} \wedge dt = dq_n \wedge dp_n + dH_n \wedge dt$, so the transformation $n \to n + 1$ is a canonical transformation on the extended phase space with coordinates (q_n, p_n, t). The generating function for this canonical transformation is

$$\mathcal{F}(q_n, q_{n+1}) = \ell \log(q_{n+1} + q_n) + \tfrac{2}{3} q_{n+1}^3 + t q_{n+1} \tag{63}$$

so that

$$p_n = -\frac{\partial \mathcal{F}}{\partial q_n}, \qquad p_{n+1} = \frac{\partial \mathcal{F}}{\partial q_{n+1}}.$$

The Bäcklund transformation formulae (62) imply that any sequence of solutions q_n of P_{II} (labelled by $n = \ell - 1/2$) satisfies

$$\frac{\ell}{q_{n+1} + q_n} + \frac{\ell - 1}{q_n + q_{n-1}} + 2q_n^2 + t = 0, \tag{64}$$

which is a discrete form of P_I, whilst p_n satisfies

$$(p_{n+1} - p_{n-1})^2 p_n^4 - 4\ell^2 (p_{n+1} + 2p_n + p_{n-1} - 2t) p_n^2 + 4\ell^4 = 0. \tag{65}$$

We will now show how the above results carry over into the discrete case.

Proposition 5.1. *In terms of the variables $q_n = (g_n + 1)/h$ and p_n, the Bäcklund transformation (38) for the alt-dP_{II} equation (35) (corresponding to $n \to n + 1$) can be written together with* *its inverse (corresponding to $n \to n-1$) as*

$$\begin{aligned}
q_{n+1} &= \frac{q_n p_n + \ell}{h(q_n p_n + \ell) - p_n},\\
p_{n+1} &= (q_n p_n + \ell)^2 \left(\frac{2}{p_n^2} - \frac{h^2}{p_n} \right) - \frac{h}{p_n}(t - 2p_n)(q_n p_n + \ell) - p_n + t; \qquad (66)\\
q_{n-1} &= -q_n + \frac{1 - \ell + hq_n\big(-2q_n^3 + q_n p_n - q_n t + \ell - 1\big) + h^2 q_n^3(-2p_n + t) + h^3 q_n^4 p_n}{-p_n + t + 2q_n^3 - hq_n\big(2q_n^2 - 3p_n + 2t\big) + h^2 q_n^2(-3p_n + t) + h^3 q_n^3 p_n},\\
p_{n-1} &= -p_n + t + 2q_n^2 + h(2p_n - t)q_n - h^2 p_n q_n^2.
\end{aligned}$$

This Bäcklund transformation is a symplectic map in the phase space with coordinates (q_n, p_n), with the generating function

$$\begin{aligned}
\mathcal{F}(q_n, q_{n+1}) = \frac{2q_{n+1}}{h^2} + \frac{2}{h^3(1 - hq_{n+1})} - \frac{1}{h^3}\left(h^2 t - 4 + \ell h^3\right) \log(1 - hq_{n+1})&\\
+ \ell \log(hq_n q_{n+1} - q_{n+1} - q_n)& \qquad (67)
\end{aligned}$$

such that $d\mathcal{F} = p_{n+1} dq_{n+1} - p_n dq_n$; in other words the canonical Poisson bracket $\{q_n, p_n\} = 1$ is preserved for $n \to n+1$. Equations (66) and the generating function reduce to those of the Bäcklund transformation for P_{II} in the continuum limit $h \to 0$.

Proof. The formulae (66) follow immediately from lemma 4.2 and corollary 4.1. To verify that the map $(q_n, p_n) \mapsto (q_{n+1}, p_{n+1})$ is symplectic, it is sufficient to calculate directly that its Jacobian determinant is equal to 1. This also follows directly from the closure of the exact one-form $d\mathcal{F}$, upon checking that $p_n = -\frac{\partial \mathcal{F}}{\partial q_n}$, $p_{n+1} = \frac{\partial \mathcal{F}}{\partial q_{n+1}}$ with the generating function $\mathcal{F}$ as in the formula (67). It is straightforward to verify that the relations (66) have the correct continuum limit given by (62), and that (67) also yields (63) when $h \to 0$. □

Remark. The symplectic structure for the *alt*-dP_{II} equation can be derived from Okamoto's Hamiltonian formulation for P_{III}, since the canonical coordinates (q_n, p_n) above are related to those for P_{III} by a shift and rescaling (cf section 2.3 in [30] for instance).

Corollary 5.1. *For the alt-dP_{II} equation (35), the analogue of (64) is the equation*

$$\frac{\ell}{q_{n+1} + q_n - hq_{n+1}q_n} + \frac{(\ell - 1)}{q_n + q_{n-1} - hq_n q_{n-1}} + \frac{2q_n^2}{1 - hq_n} + t - h\ell = 0. \qquad (68)$$

In terms of g_n this is the second-order difference equation

$$\frac{\ell}{g_{n+1}g_n - 1} + \frac{(\ell - 1)}{g_n g_{n-1} - 1} + \frac{2}{h^3}\left(g_n + \frac{1}{g_n}\right) + \ell + \frac{4}{h^3} - \frac{t}{h} = 0, \qquad (69)$$

which is another form of the alt-dP_{II} equation. The conjugate momentum p_n satisfies a third-order recurrence relation in n, namely

$$\begin{aligned}
p_{n+1}&\left\{ \frac{p_n}{2n+1} - \frac{hp_n\big[(2n-1)^2 + 2p_{n-1}^2(p_n - p_{n-2})\big]}{4p_{n-1}(4n^2 - 1)} - \frac{h^2 p_n t}{4(2n+1)} + \frac{h^3 p_n(2n-1)}{8(2n+1)} \right\}\\
&+ p_{n-2}\left\{ -\frac{p_{n-1}}{2n-1} - \frac{hp_{n-1}\big[(2n+1)^2 + 2p_n^2(p_{n-1} - p_{n+1})\big]}{4p_n(4n^2 - 1)} \right.\\
&\left. + \frac{h^2 p_{n-1} t}{4(2n-1)} + \frac{h^3 p_{n-1}(2n+1)}{8(2n-1)} \right\} + \frac{h^2 p_n^2 p_{n-1}^2}{2(4n^2 - 1)} + \frac{h(4n^2 - 1)}{8p_n p_{n-1}}\\
&- \frac{(2n+1)[h^3(2n-1) - 2h^2 t + 8]}{16p_n} - \frac{(2n-1)[h^3(2n+1) + 2h^2 t - 8]}{16p_{n-1}}
\end{aligned}$$

$$+ p_n p_{n-1}\left[\frac{8-2h^2t+(4n^2-3)h^3}{4(4n^2-1)}\right] - \frac{(6n+5)hp_n}{4(2n+1)} - \frac{(6n-5)hp_{n-1}}{4(2n-1)}$$
$$+\frac{h}{32}[32t+16h-4h^2t^2-4h^3t+(4n^2-1)h^4]=0. \tag{70}$$

Proof. Upon solving the first of (66) for p_n and substituting in the last one, (68) results. One can eliminate the quadratic terms in q_n from the second and fourth of (66) to get

$$q_n = -\frac{2n+1}{4p_n} - \frac{2(p_{n+1}-p_{n-1})p_n + (t-2p_n)(2n+1)h}{2(2n+1)(h^2p_n-2)}. \tag{71}$$

After substituting this into the first of (66) with $n\to n-1$, we obtain (70). □

Remarks. The fact that the alt-dP_{II} equation is self-dual, in the sense that the superposition formula (69) for its Bäckund transformation is (up to rescaling and reversing the roles of the dependent variable and the Bäcklund parameter) the equation itself, was first noted in [12]. Equation (37) in [32] is the continuum limit of equation (70). The latter relation allows one to obtain p_{n+1} uniquely given p_n, p_{n-1}, p_{n-2}. Remembering that $p_n = \frac{\tau_{n-1}\tau_{n+1}}{2\tau_n^2}$, we see that if $\tau_j(t,h)$ for $j=n-3,n-2,\ldots,n+1$ are given for a particular value of t, then τ_{n+2} can be evaluated for this same value of t. More precisely, (70) is equivalent to a recurrence relation for τ_n that involves no shifts in t and is linear in τ_{n+2} and τ_{n-3}; in particular this relation, which is of fifth order in n, is satisfied by the discrete Yablonskii–Vorob'ev polynomials Y_n for $n\in\mathbb{Z}$. The latter recurrence for the tau-functions, which is omitted here, tends to equation (38) in [32] in the continuum limit. It is also possible to write q_n in terms of unshifted τ_n, by substituting the right-hand side of (47) for p_n in (71).

We now consider the map in the (q_n, p_n) phase space corresponding to shifting in t rather than n.

Proposition 5.2. *In the phase space with coordinates (q_n, p_n), the shift $t\to t+h$ corresponding to the alt-dP_{II} system (36) is given by*

$$\overline{q}_n = \frac{-2q_n + h(t-2p_n) + h^2p_nq_n}{-2-2hq_n+h^2(t-p_n)+h^3q_np_n},$$
$$\overline{p}_n = \frac{1}{(h^2p_n-2)^2}(-2-2hq_n+h^2(t-p_n)+h^3q_np_n)$$
$$\times(-2p_n - 2h(\ell+q_np_n)+h^2p_n(t-p_n)+h^3p_n(\ell+q_np_n)). \tag{72}$$

This is a symplectic map with the generating function

$$S = \frac{1}{h^3}(4-h^2t)\log(2-h^2p_n) + \frac{\ell}{2}\log\frac{\overline{p}_n}{p_n} + \frac{1}{h}(p_n-\overline{p}_n)$$
$$+\frac{1}{h}\sqrt{\ell^2h^2+4p_n\overline{p}_n} - \ell\tanh^{-1}\left(\frac{\sqrt{\ell^2h^2+4p_n\overline{p}_n}}{\ell h}\right) \tag{73}$$

such that $dS = q_n dp_n - \overline{q}_n d\overline{p}_n$.

Proof. Equations (72) just correspond to rewriting (36) in terms of $q_n = (g_n+1)/h, \overline{q}_n = (\overline{g}_n+1)/h$. It is extremely easy to check directly from (36) that the symplectic form $\omega_n = dq_n\wedge dp_n = \frac{1}{h}dg_n\wedge dp_n$ is preserved by the shift in t. To find the generating function, it is convenient to write $g_n, \overline{g}_n$ in terms of $p_n, \overline{p}_n$, giving

$$\frac{\partial\hat{S}}{\partial p_n} = g_n = -\left(\frac{4-h^2t}{2-h^2p_n}\right) - \frac{1}{2p_n}\left(\ell h \mp \sqrt{\ell^2h^2+4\overline{p}_np_n}\right),$$
$$\frac{\partial\hat{S}}{\partial\overline{p}_n} = -\overline{g}_n = \frac{1}{2\overline{p}_n}\left(\ell h \pm\sqrt{\ell^2h^2+4\overline{p}_np_n}\right). \tag{74}$$

(One has to take the upper choice of sign in each case to get the correct continuum limit.) Having found an $\hat{S}$ for which the relations (74) hold, $S = (\hat{S} + p_n - \overline{p}_n)/h$ provides the generating function in (73). □

Corollary 5.2. *The discrete P_{XXXIV} equation associated with the alt-dP_{II} equation (35) can be written either as*

$$\pm\sqrt{\ell^2h^2 + 4p_n\overline{p}_n} \pm \sqrt{\ell^2h^2 + 4p_n\underline{p}_n} = \frac{2p_n(4 - h^2t)}{2 - h^2p_n}, \tag{75}$$

or with the square root signs removed as

$$(\ell^2h^2 + 4p_n\overline{p}_n)(\ell^2h^2 + 4p_n\underline{p}_n) = \left(\frac{2p_n^2(4 - h^2t)^2}{(2 - h^2p_n)^2} - \ell^2h^2 - 2p_n(\overline{p}_n + \underline{p}_n)\right)^2.$$

Proof. The second-order recurrence relation for p_n is obtained by downshifting the second of (74) and equating it to minus the first. □

Remark. The 3-point correspondence (75) is equivalent to an analogous equation for the variable y_n, that is equation (5.3) in [12], and equation (75) has the same structure as certain discrete Ermakov–Pinney equations constructed in [33]. The presence of this structure is due to the connection with discrete Schrödinger equations (for which, see the proof of proposition 6.1).

6. Lax pair for *alt-dP$_{II}$*

The continuum P_{II} is equivalent to the pair of bilinear equations (30). A Lax pair for P_{II} is given by the linear problem

$$\frac{\partial\Psi}{\partial t} = \mathcal{B}\Psi, \qquad \frac{\partial\Psi}{\partial\eta} = \mathcal{A}\Psi, \tag{76}$$

where

$$\mathcal{B} = \begin{pmatrix} -\dfrac{\eta}{2} & -\dfrac{\mathrm{i}\tau_{n+1}}{2\tau_n} \\ \dfrac{\mathrm{i}\tau_{n-1}}{2\tau_n} & \dfrac{\eta}{2} \end{pmatrix} \tag{77}$$

and

$$\mathcal{A} = \begin{pmatrix} -\eta^2 - \dfrac{t}{2} + \dfrac{\tau_{n-1}\tau_{n+1}}{2\tau_n^2} & -\dfrac{\mathrm{i}\eta\tau_{n+1}}{\tau_n(t)} + \mathrm{i}\dfrac{d}{dt}\left(\dfrac{\tau_{n+1}}{\tau_n}\right) \\ \dfrac{\mathrm{i}\eta\tau_{n-1}}{\tau_n} + \mathrm{i}\dfrac{d}{dt}\left(\dfrac{\tau_{n-1}}{\tau_n}\right) & +\eta^2 + \dfrac{t}{2} - \dfrac{\tau_{n-1}\tau_{n+1}}{2\tau_n^2} \end{pmatrix}. \tag{78}$$

Consistency of (76) requires

$$\frac{\partial\mathcal{B}}{\partial\eta} - \frac{\partial\mathcal{A}}{\partial t} + [\mathcal{B}, \mathcal{A}] = 0, \tag{79}$$

leading to the two conditions

$$\ddot{\phi}_\pm + V\phi_\pm = 0, \tag{80}$$

where we have set

$$\phi_\pm = \mp\frac{\mathrm{i}\tau_{n\pm1}}{2\tau_n}, \qquad V = \frac{t}{2} - 2\phi_+\phi_-.$$

The choice of gauge $V = t/2 - 2\phi_+\phi_- = 2\frac{d^2}{dt^2}\log\tau_n$ gives precisely equation (34), and with this normalization for the tau-functions the conditions (80) give the first equation in (30) for $F = 0$, together with the same equation for $n \to n+1$. The bilinear equation (33) is a consequence, and the second equation in (30) then follows. It is well known that (79) is an isomonodromy condition: the monodromy of the solutions of the second linear equation (76) in the complex η plane is independent of t if and only if P_{II} holds.

In the discrete case the situation is completely analogous, based on a linear problem that comes from the first part of the discrete Toda Lax pair (5).

Proposition 6.1. *The linear problem*

$$\overline{\Psi} = B\Psi, \qquad \lambda\partial_\lambda\Psi = A\Psi \tag{81}$$

where

$$B = \begin{pmatrix} \dfrac{1}{\lambda} & \dfrac{h\tau_{n+1}}{2\mathrm{i}\tau_n} \\ -\dfrac{h\tau_{n-1}}{2\mathrm{i}\tau_n} & \lambda \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\lambda} & \phi_+ \\ \phi_- & \lambda \end{pmatrix} \tag{82}$$

and

$$A = \begin{pmatrix} \dfrac{2}{h^3}\left(-\dfrac{1}{\lambda^2} + 2(\phi_+\underline{\phi}_- + 1) - \lambda^2\right) - \dfrac{t}{h} + \dfrac{1}{2} & \dfrac{4}{h^3}\left(-\dfrac{\phi_+}{\lambda} + \underline{\phi}_+\lambda\right) \\ \dfrac{4}{h^3}\left(-\dfrac{\underline{\phi}_-}{\lambda} + \phi_-\lambda\right) & \dfrac{2}{h^3}\left(\dfrac{1}{\lambda^2} - 2(\underline{\phi}_+\phi_- + 1) + \lambda^2\right) + \dfrac{t}{h} - \dfrac{1}{2} \end{pmatrix}$$

constitutes a Lax pair for the alt-dP_{II} equation (35), which is equivalent to the consistency condition

$$\lambda\partial_\lambda B + BA - \overline{A}B = 0 \tag{83}$$

for (81). In the limit $h \to 0$, this linear system reduces to the Lax pair (76) for P_{II}.

Proof. The consistency condition (83) implies the two relations

$$\overline{\phi}_\pm + V\phi_\pm + \underline{\phi}_\pm = 0, \tag{84}$$

which take the form of discrete Schrödinger equations with

$$V = \frac{(h^2t-4)}{2(1-\phi_+\phi_-)} = \frac{2\tau_n^2(h^2t-4)}{4\tau_n^2 - h^2\tau_{n+1}\tau_{n-1}}.$$

If we fix the gauge so that $V = -2\tau_n^2/(\overline{\tau}_n\underline{\tau}_n)$, then we get precisely the relation (49), and upon substituting the latter expression for V into each of the equations (84) in turn we find that (45) and (48) hold. Then by theorem 4.1, $g_n = -\underline{\tau}_{n-1}\tau_n/(\tau_{n-1}\underline{\tau}_n)$ satisfies the *alt-dP_{II}* equation. For the continuum limit one should take $\lambda = \mathrm{e}^{\frac{h\eta}{2}}$, which gives

$$A = \frac{2}{h}\mathcal{A} + \mathcal{O}(h^0), \qquad B = 1 + h\mathcal{B} + \mathcal{O}(h^2), \tag{85}$$

so that the condition (79) arises from (83) as $h \to 0$. □

Remark. In [12] a different 2×2 Lax pair is presented for the *alt-dP_{II}* equation, by reduction from the modified Boussinesq lattice. However, we have not found a direct relationship between these two Lax pairs.

7. Concluding remarks

There are many ways to construct a discretization of a given integrable differential equation, depending on which particular properties (e.g. Lax pair, explicit solutions, Poisson structure, Hirota bilinear form, ...) one most wishes to preserve. (For a thorough account of the Hamiltonian approach, see [34].) Due to the non-uniqueness of discretization, it is not always clear what is the 'best' discrete analogue of a continuous system. The derivation of the $alt\text{-}dP_{II}$ equation presented here was initially motivated by the construction of exact rational solutions, but it has turned out that analogues of all the other structures associated with the second Painlevé equation arise naturally here as well. The fact that this discretization scheme for P_{II}, based on the discrete Toda lattice, turned out to be connected with the superposition formula for P_{III} was completely unexpected by us, but led to different expressions for the polynomial tau-functions in terms of Jacobi–Trudi determinants. In [12] other solutions of $alt\text{-}dP_{II}$ were constructed in terms of Casorati determinants of discrete Airy functions. In future we would like to analyse other solutions of equation (35). We constructed the rational solutions of this equation from polynomial tau-functions given by Hankel determinants, but recent results for the continuous case [7] lead us to expect that all tau-functions should have a similar structure.

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Appendix

Here we present the proof of theorem 4.4. The result essentially follows from the fact that the Bäcklund transformation for P_{III} generates a sequence of rational solutions w_n given by (56), which simultaneously provide rational solutions of the $alt\text{-}dP_{II}$ equation (53) by setting $g_n = -1/w_n$ with $z_n = n + 1/2$ for $n \in \mathbb{Z}$ and $\mu = -\nu$. On the other hand, theorem 4.2 and corollary 5.1 together imply that the $alt\text{-}dP_{II}$ equation in the form (69) has rational solutions, given by suitable ratios of discrete Yablonskii–Vorob'ev polynomials, when $\ell = n + 1/2$ with $n \in \mathbb{Z}$. Upon comparing (53) with (69) we see that these two sets of rational solutions coincide if we identify $x = 4/h^3$, $\nu = t/h - 4/h^3$, and then it follows from theorem 4.3 and theorem 4.2 that

$$w_n = \frac{Z_{n-1}(t,h)Z_n(t-h,h)}{Z_{n-1}(t-h,h)Z_n(t,h)} = \frac{Y_{n-1}(t,h)Y_n(t-h,h)}{Y_{n-1}(t-h,h)Y_n(t,h)} \tag{A.1}$$

for all $n \in \mathbb{Z}$, where $Z_n(t,h)$ denotes the right-hand side of (57) for $n \geqslant 0$, and the identity extends to negative n upon setting $Z_{-n} = Z_{n-1}$. Then it is necessary to show that $Y_n(t,h) = Z_n(t,h)$ for all n. It is sufficient to consider $n \in \mathbb{N}$ as in theorem 4.4, as the extension to negative n is trivial.

For $n = 0$ the result $Y_0 = 1 = Z_0$ is obvious, so to use induction we assume that $Y_{N-1} = Z_{N-1}$ and then from (A.1) with $n = N$ it holds that

$$\frac{Z_N(t-h,h)}{Z_N(t,h)} = \frac{Y_N(t-h,h)}{Y_N(t,h)}. \tag{A.2}$$

Both sides of the relation (A.2) are ratios of polynomials in t (for Z_N this follows from the determinant formula for $\mathcal{D}_N$), and the rational functions on each side must have the same zeroes and poles. However, although theorem 3.1 implies that the numerator and denominator on the right-hand side have no common factors, we cannot immediately assert that the same is true on the left-hand side without knowing the degree of $\mathcal{D}_N(x,\nu)$ in ν. (The proof of corollary 2 in [13] just gives the degree in x of $\mathcal{D}_N$, denoted F_N there, as $N(N+1)/2$.) However, by proposition 3 in [13] these Jacobi–Trudi determinants satisfy the recurrence

$$(2n+1)\mathcal{D}_{n+1}\mathcal{D}_{n-1} + x(\mathcal{D}_n\mathcal{D}_n'' - (\mathcal{D}_n')^2) + \mathcal{D}_n\mathcal{D}_n' - (x+\nu)\mathcal{D}_n^2 = 0, \qquad \text{(A.3)}$$

with $\mathcal{D}_{-1} = 1 = \mathcal{D}_0$ and $'$ denoting d/dx. In fact, this recurrence can be used to show that $\mathcal{D}_N(x,\nu)$ is also of degree $N(N+1)/2$ in ν, but we do not need this. Instead, setting $\nu = 0$ in (A.3) leads to the expression $\mathcal{D}_n(x,0) = c_n^{-1}x^{n(n+1)/2}$ where c_n is given in terms of double factorials by (58). Hence for the case at hand we have $Z_N(4/h^2,h) = c_n h^{n(n+1)/2}\mathcal{D}_n(4/h^3,0) = (4/h^2)^{n(n+1)/2} = Y_N(4/h^2,h)$ by lemma 3.1. Using (A.2) it then follows by induction that $Z_N(4/h^2+kh,h) = Y_N(4/h^2+kh,h)$ for all $k\in\mathbb{Z}$, and since these two polynomials agree for infinitely many values of t they must be equal, as required.

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